On logarithm

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Laws of Logarithm (Easier)

1. $\log_a N = x \Leftrightarrow N = a^x$, where $N > 0$, $a > 0$, $a \neq 1$.

In particular, $\log_{10} N = x \Leftrightarrow N = 10^x$

2. $x = a^{\log_a x}$, $\log_a a^x = x$

3. $\log_a 1 = 0$, $\log_a a = 1$

4. $\log_a MN = \log_a M + \log_a N$, where $M , N > 0$

5. $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

6. $\log_a N^m = m \log_a N$, where m is rational. (in particular, $\log_a \sqrt{N} = \dfrac{1}{2}\log_a N$)

Laws of Logarithm (Harder)

1. Change of base : $\log_a b = \dfrac{\log_c b}{\log_c a}$, $\log_a b = \dfrac{\log_a x}{\log_b x}$

2. $\log_a b = \dfrac{1}{\log_b a}$

3. $\log_a b \times \log_b c = \log_a c$, $\log_a b \times \log_b c \times \log_c d = \log_a d$

4. $\log_{a^n} b^m = \dfrac{m}{n}\log_a b$, in particular, $\log_{a^n} b^n = \log_a b$, $\log_{\sqrt{a}} \sqrt{b} = \log_a b$

5. $\log_{\frac{1}{a}} b = -\log_a b = \log_a\left(\dfrac{1}{b}\right)$

6. $b^{\log a} = a^{\log b}$, where the logarithms are in the same base.

Example 1

Solve for x : $\log_{10} x^2 = 4$

Solution (Mistake)

$$\log_{10} x^2 = 4 \quad \Rightarrow \quad 2\log_{10} x = 4 \quad \Rightarrow \log_{10} x = 2 \quad \Rightarrow x = 10^2 = 100$$

(Correct)

$$\log_{10} x^2 = 4 \quad \Rightarrow \quad x^2 = 10^4 = 10000 \quad \Rightarrow x = \pm\sqrt{10000} \quad \Rightarrow x = \pm 100$$

Example 2

Given that a, b, c are positive numbers greater than 1, simplify :

$$\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc}$$

Solution

$$\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc} = \frac{\log_a a}{\log_a abc} + \frac{\log_b b}{\log_b abc} + \frac{\log_c c}{\log_c abc} = \log_{abc} a + \log_{abc} b + \log_{abc} c$$

$$= \log_{abc} abc = \underline{\underline{1}}$$

Example 3

Let $a = \log_2 3$, $b = \log_3 7$. Express $\log_{42} 56$ in terms of a, b .

Solution

$$\log_{42} 56 = \frac{\log_2 56}{\log_2 42} = \frac{\log_2 2^3 \times 7}{\log_2 2 \times 3 \times 7} = \frac{3 + \log_2 7}{1 + \log_2 3 + \log_2 7} = \frac{3 + \log_2 3 \log_3 7}{1 + \log_2 3 + \log_2 3 \log_3 7} = \frac{3 + ab}{1 + a + ab}$$

Example 4

If $xy^{p-1} = a$, $xy^{q-1} = b$ and $xy^{r-1} = c$, prove that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$

Solution

$xy^{p-1} = a \qquad \Rightarrow \log x + (p-1)\log y = \log a \qquad$ …. (1)

$xy^{q-1} = b \qquad \Rightarrow \log x + (q-1)\log y = \log b \qquad$ …. (2)

$xy^{r-1} = c \qquad \Rightarrow \log x + (r-1)\log y = \log c \qquad$ …. (3)

$(1) \times (q-r)$, $\qquad (q-r)\log x + (q-r)(p-1)\log y = (q-r)\log a \qquad$ …. (4)

$(2) \times (r-p)$, $\qquad (r-p)\log x + (r-p)(q-1)\log y = (r-p)\log b \qquad$ …. (5)

(3) × (p – q), $(p-q)\log x+(p-q)(r-1)\log y=(p-q)\log c$ …. (6)

(4) + (5) + (6), $(q-r)\log a+(r-p)\log b+(p-q)\log c=0$

Example 5

If $\log_a n=x$ and $\log_c n=y$ where $n\neq 1$, prove that $\dfrac{x-y}{x+y}=\dfrac{\log_b c-\log_b a}{\log_b c+\log_b a}$.

Solution

$$\frac{x-y}{x+y}=\frac{\log_a n-\log_c n}{\log_a n+\log_c n}=\frac{\dfrac{\log_b n}{\log_b a}-\dfrac{\log_b n}{\log_b c}}{\dfrac{\log_b n}{\log_b a}+\dfrac{\log_b n}{\log_b c}}=\frac{\dfrac{1}{\log_b a}-\dfrac{1}{\log_b c}}{\dfrac{1}{\log_b a}+\dfrac{1}{\log_b c}}=\frac{\dfrac{\log_b c-\log_b a}{\log_b c\log_b a}}{\dfrac{\log_b c+\log_b a}{\log_b c\log_b a}}=\frac{\log_b c-\log_b a}{\log_b c+\log_b a}$$

Example 6

Solve the equation $\log_x 8+\log_8 x=\dfrac{13}{6}$

Solution

$$6\log_x 8+6\log_8 x=13$$

$$\frac{6}{\log_8 x}+6\log_8 x=13$$

$$6(\log_8 x)^2-13\log_8 x+6=0$$

$$(2\log_8 x-3)(3\log_8 x-2)=0$$

$\log_8 x=\dfrac{3}{2}$ or $\log_8 x=\dfrac{2}{3}$

$x=8^{3/2}$ or $x=8^{2/3}$

$\therefore$ $x=16\sqrt{2}$ or 4

Exercise

1. Solve $\begin{cases}\log x:\log y:\log z=1:2:(-1)\\ xyz=100\end{cases}$ for x , y , z .

2. If a, b, c, d > 0, show that $\log\left(\dfrac{a}{b}\right)\log\left(\dfrac{c}{d}\right)=\log\left(\dfrac{a}{c}\right)\log\left(\dfrac{b}{d}\right)+\log\left(\dfrac{a}{d}\right)\log\left(\dfrac{c}{b}\right)$,

where the logarithms are all of the same base.

3. Given $a = \log_9 16$, express $\log_6 \sqrt{24}$ in terms of a .

4. If $x = \log_a\left(\frac{9}{8}\right)$, $y = \log_a\left(\frac{16}{15}\right)$ and $z = \log_a\left(\frac{24}{25}\right)$, prove that $3x + 4y – 2z = \log_a 2$,

$5x + 6y – 3z = \log_a 3$ and $7x + 9y – 5z = \log_a 5$. If $a = 10$, find without using calculator, the value of $20x + 26y – 14z$.

5. If $\log_a x + \log_b y = \log_a y + \log_b x$, show that $a = b$ or $x = y$.

Answers:

1. $\begin{cases} \log x : \log y : \log z = 1 : 2 : (-1) \quad ....(1) \\ \quad xyz = 100 \quad ....(2) \end{cases}$

From (1), $\log x = k$, $\log y = 2k$, $\log z = k$ $\Rightarrow$ $x = 10^k$, $y = 10^{2k}$, $z = 10^{-k}$ …. (3)

$\therefore$ $xyz = 10^{k+2k-k} = 10^{2k} = 10^2$, from (2)

$\therefore$ $2k = 2$ and $k = 1$.

From (3), $x =10$, $y = 100$, $z = 0.1$.

2. $\log\left(\frac{a}{b}\right)\log\left(\frac{c}{d}\right)$ = [log a – log b] [log c – log d] = log a log c – log a log d – log b log c + log b log d

$\log\left(\frac{a}{c}\right)\log\left(\frac{b}{d}\right) + \log\left(\frac{a}{d}\right)\log\left(\frac{c}{b}\right)$ = [log a – log c][log b – log d] + [log a – log d][log c – log b]

= log a log c – log a log d – log b log c + log b log d

3. $a = \log_9 16 \Rightarrow a = \frac{\log_2 2^4}{\log_2 3^2} \Rightarrow \frac{2}{a} = \log_2 3$

$$\log_6 \sqrt{24} = \frac{1}{2}\left(\frac{\log_2 2^3 + \log_2 3}{\log_2 3 + \log_2 2}\right) = \frac{3 + \log_2 3}{2(\log_2 3 + 1)} = \frac{3 + (2/a)}{2((2/a) + 1)} = \frac{3a + 2}{2a + 4}$$

4. $3x + 4y - 2z = 3\log_a\left(\frac{9}{8}\right) + 4\log_a\left(\frac{16}{15}\right) - 2\log_a\left(\frac{24}{25}\right) = \log_a\left(\frac{3^2}{2^3}\right)^3\left(\frac{2^4}{3\times 5}\right)^4\left(\frac{2^3\times 3}{5^2}\right)^{-2} = \log_a 2$

$5x + 6y - 3z = 5\log_a\left(\frac{9}{8}\right) + 6\log_a\left(\frac{16}{15}\right) - 3\log_a\left(\frac{24}{25}\right) = \log_a\left(\frac{3^2}{2^3}\right)^5\left(\frac{2^4}{3\times 5}\right)^6\left(\frac{2^3\times 3}{5^2}\right)^{-3} = \log_a 3$

$7x + 9y - 5z = 7\log_a\left(\frac{9}{8}\right) + 9\log_a\left(\frac{16}{15}\right) - 5\log_a\left(\frac{24}{25}\right) = \log_a\left(\frac{3^2}{2^3}\right)^7\left(\frac{2^4}{3\times 5}\right)^9\left(\frac{2^3\times 3}{5^2}\right)^{-5} = \log_a 5$

$20x + 26y - 14z = 20\log_a\left(\frac{9}{8}\right) + 26\log_a\left(\frac{16}{15}\right) - 14\log_a\left(\frac{24}{25}\right) = \log_a\left(\frac{3^2}{2^3}\right)^{20}\left(\frac{2^4}{3\times 5}\right)^{26}\left(\frac{2^3\times 3}{5^2}\right)^{-14}$

$= \log_a 100 = \log_{10} 100 = \underline{\underline{2}}$

5. $\log_a x + \log_b y = \log_a y + \log_b x$

$\log_a x + \frac{\log_a y}{\log_a b} = \log_a y + \frac{\log_a x}{\log_a b}$

$\log_a b \log_a x + \log_a y = \log_a b \log_a y + \log_a x$

$(\log_a b - 1)(\log_a x - \log_a y) = 0$

$\log_a b = 1$ or $\log_a x = \log_a y$ $\Rightarrow$ $a = b$ or $x = y$.